POSITIVE MASS CONJECTURE FOR FIVE DIMENSIONAL LORENTZIAN MANIFOLDS

by

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1 Introduction

Let N be an $(n+1)$-dimensional Lorentzian manifold with Lorentzian metric $\widetilde{g}$ of signature $(-1, 1, \cdots, 1)$, which satisfies the Einstein equations

$$\widetilde{R}_{\alpha\beta} - \frac{\widetilde{R}}{2}\,\widetilde{g}_{\alpha\beta} = T_{\alpha\beta}, \tag{1.1}$$

where $\widetilde{R}_{\alpha\beta}$, $\widetilde{R}$ are the Ricci and scalar curvatures of $\widetilde{g}$ respectively, $T_{\alpha\beta}$ is a symmetric tensor field which is interpreted physically as the energy-momentum tensor of matter. Choosing an orthonormal frame $\{e_\alpha\}$ with e_0 timelike. Then, physically, T_{00} is interpreted as the local mass density, and T_{0i} is interpreted as the local angular momentum.

Definition 1.1 *A spacelike hypersurface M of N is called asymptotically flat of order τ if there is a compact set $K \subset M$ such that $M - K$ is the disjoint union of a finite number of subsets $M_1, \cdots, M_k$ – called the "ends" of M – each diffeomorphic to the complement of a contractible compact set in R^n. Under the diffeomorphism the metric of $M_l \subset N$ is of the form*

$$g_{ij} = \delta_{ij} + a_{ij} \tag{1.2}$$

in the standand coordinates $\{x^i\}$ on R^n, where

$$\begin{aligned} a_{ij} &= O(r^{-\tau}) \\ \partial_k a_{ij} &= O(r^{-\tau-1}) \\ \partial_l \partial_k a_{ij} &= O(r^{-\tau-2}). \end{aligned} \tag{1.3}$$

Furthermore, the second fundamental forms of M satisfy

$$\begin{aligned} h_{ij} &= O(r^{-\tau-1}) \\ \partial_k h_{ij} &= O(r^{-\tau-2}). \end{aligned} \tag{1.4}$$

We will often identify the end $M_l \subset M$ with the corresponding set $M_l \subset R^n$.

For spacelike asymptotically flat hypersurface M, we can define the total energy and the total momentum. These quantities include contributions from both the matter and the gravitational field itself. They are defined in each asymtotic end M_l as limits over the sphere $S_{R,l}$ of radius R in $M_l \subset R^n$.

Definition 1.2 *Total energy of end M_l is defined as*

$$E_l = \lim_{R\to\infty} \frac{1}{4(n-1)\omega_{n-1}} \int_{S_{R,l}} (\partial_j g_{ij} - \partial_i g_{jj})\, d\Omega^i. \tag{1.5}$$

Total momentum of end M_l is defined as

$$p_{lk} = \lim_{R\to\infty} \frac{1}{4(n-1)\omega_{n-1}} \int_{S_{R,l}} 2(h_{ik} - \delta_{ik}h_{jj})d\Omega^i. \tag{1.6}$$

When the asymptotic order $\tau > \frac{n-1}{2}$, these quantities are finite, also R. Bartnik [B1] showed that E_l is independent on the choice of asymptotic coordinates.

Physically, that M has nonnegative local mass density is interpreted as the dominant energy condition [H-E]. The mathematical definition is

Definition 1.3 *M is satisfied the dominant energy condition if for each point $p \in M$ and for each timelike vector e_0 at p, $T(e_0,\ e_0) \geq 0$ and $T(e_0,\ )$ is a non-spacelike covector. This has the following consequences: if $\{e_\alpha | \alpha = 0, 1, \cdots, n\}$ is an adapted orthonormal frame field at $p \in M$ with e_0 normal to M and $e_1, \cdots, e_n$ tangent to M, then*

$$\begin{aligned} T^{00} &\geq |T^{\alpha\beta}|, \\ T^{00} &\geq (-T_{0i}T^{0i})^{\frac{1}{2}}. \end{aligned} \tag{1.7}$$

(Here, and henceforth, repeated indices are summed with Latin indices running from 1 to n and Greek indices running from 0 to n.)

In general relativity, N is 4-dimensional spacetime manifold, a gravitational system with nonegative matter density should have nonnegative total energy. However, the total energy is defined by a nonlinear process, it makes the problem unclear and nontrivial. This is called the positive mass conjecture. Here, we give the n-dimensional statement of the positive mass conjecture.

Possitive Mass Conjecture I *Let N be an $(n+1)$-dimensional Lorentzian manifold with Lorentzian metric $\tilde{g}$ of signature $(-1, 1, \cdots, 1)$, $M \subset N$ be an n-dimensional space-like asymptotically flat hypersurface of order $\tau > \frac{n-1}{2}$. If the dominant energy condition holds on M, then, on each end M_l,*

$$E_l \geq |P_l| \equiv (\sum_{k=1}^{n} p_{lk}^2)^{\frac{1}{2}}.$$

If $E_{l_0} = 0$ for some l_0, then M has only one end and N is flat along M.

The Gauss and Codazzi equations for $M \subset N$ give that (see §2)

$$\begin{aligned} T_{00} &= \frac{1}{2}\left(R - \sum h_{ij}^2 + H^2\right) \\ T_{0i} &= \nabla_j h_{ji} - \nabla_i H, \end{aligned}$$

where R and $H = \sum h_{ii}$ are saclar curvature and mean curvature of M respectively. In the case of maximal spacelike hypersurface, i.e., $H = 0$, the dominant energy condition implies $R \geq 0$. Hence the positive mass conjecture, in the case, states that

Possitive Mass Conjecture II *Let M be a n-dimensional asymptotically flat manifold of order $\tau > \frac{n-1}{2}$. If the scalar curvature $R \geq 0$, then, on each end M_l, $E_l \geq 0$. If $E_{l_0} = 0$ for some l_0, then M is isomorphic to R^n.*

When $n = 3$, the positive mass conjecture was originally conjectured more than thirty years ago by Arnowitt, Deser and Misner [A-D-M]. Subsequently, a great many people worked on this problem and proved various special cases. In 1978, Schoen and Yau used a geometrical method to prove this conjecture for the case of maximal spacelike hypersurface (Conjecture II) [S-Y1]. Using an auxiliary equation introduced by Jang [J], they generalized their proof to the non-maximal spacelike hypersurface case (Conjecture I) [S-Y4], and finally solving this long-standing problem. They have also applied their method to proof the positive action conjecture [S-Y3]. Two years later, Witten presented a simple new proof of the Conjecture I by using spinors although several points of his argument come from physical intuition and require justification [W1]. Soon later, Parker and Taubes gave a complete, rigorous and self-contained proof of the Conjecture I, based on Witten's formulation [P-T].

It should be pointed out much information is hinted in Schoen and Yau's approach, which is still unknown. Roughly speaking, it asserts how to understand the Einstein field equations in the "mirror" Riemannian manifold of Spacetime. The black hole appears in their proof natually. Perhaps, some of their ideas may be used to understand the recent progresses in String theory.

For the higher dimensional positive mass conjecture, only the maximal hypersurface case (Conjecture II) has been considered: Schoen gave a detail n-dimensional proof of his work with Yau which proved the Conjecture II through the use of volume minimizing hypersurface [Sc]. The proof they gave workes for $n \leq 7$ in which dimensions they have complete regularity of volume minimizing hypersurfaces. Bartnik showed the proof of Conjecture II for n-dimensional spin manifolds following Witten's approach [B1]. But nothing appears for the non-maximal hypersurface case (Conjecture I). The difficulty to generalize Witten's approach to higher dimensional Lorentzian manifold is that spinors under the non-positive define metric is far from understanding.

Here, we shall give the proof of Conjecture I for 4-dimensional spin spacelike hypersurface M in 5-dimensional Lorentzian manifold N. We define the hypersurface spinors and the hypersurface Dirac operator acts on this spinors along M in terms of $HU(1,1) \cong Spin^0(4,1)$ structure. We also derive a relation between the hypersurface Dirac operator and usually Dirac operator of M. By this relation, we derive two Weitzenböck formulas for the hypersurface Dirac operator which one was given by Witten [W1, P-T], and simplify the original argument for the existence of the hypersurface Dirac equation given by Parker and Taubes [P-T]. We prove the Conjecture I in the case. We also investigate some basic facts on $Spin^c$ structure on 4-dimensional manifolds, and define the hypersurface $Spin^c$ structure. For complete Riemannian 4-dimensional manifold with Ricci curvature bounded from below and nonnegative scalar curvature, we show that any C^2-solution of Seiberg-Witten equations is reducible. This generalizes an observation of Witten on finite energy solutions on R^4 [W2]. The motivation is to try finding some relations between the positive mass conjecture and the Seiberg-Witten theory. They are still unclear. Finally, we show that, in general, the mean curvature of M must vanish at some points when the nontrivial solutions of $\widetilde{D}\phi = 0$ exist. It is at least two aspects of importance: The nonexistence of constant mean curvature hypersurfaces which are much interesting in general relativity; on the other hand, the compactness of the moduli space of hypersurface Seiberg-Witten equations (6.13) fails since it depends on the nonzero lower bound of $|H|$ [Z]. Such a bound does not exist on the set of irreducible solutions.

2 Preliminaries

In this section, we shall study the structural equations for Lorentzian manifolds and derive the Gauss and Codazzi curvature equations for spacelike hypersurface.

Let N be an $(n+1)$-dimensional manifold with Lorentzian metric $\widetilde{g}$ of signature $(-1, 1, \cdots, 1)$. Let $\{e_\alpha\}$ be local orthonormal frame field in N, and $\{\omega^\alpha\}$ be its dual frame field so that $\widetilde{g} = -\omega_0^2 + \sum_i \omega_i^2$. The Lorentzian connection forms $\omega_{\alpha\beta}$ of N are uniquely determined by the equations

$$\begin{aligned} d\omega_0 &= \sum_i \omega_{0i} \wedge \omega_i, \\ d\omega_i &= -\omega_{i0} \wedge \omega_0 + \sum_j \omega_{ij} \wedge \omega_j, \\ \omega_{\alpha\beta} + \omega_{\beta\alpha} &= 0. \end{aligned} \tag{2.1}$$

The covariant derivatives are determined by the following equations

$$\begin{aligned} De_0 &= \sum_i \omega_{0i} e_i, \\ De_i &= \sum_j \omega_{ij} e_i - \omega_{i0} e_0. \end{aligned} \tag{2.2}$$

The curvature forms $\widetilde{\Omega}_{\alpha\beta}$ of N are given by

$$\widetilde{\Omega}_{0i} = d\omega_{0i} - \sum_k \omega_{0k} \wedge \omega_{ki},$$

$$\begin{aligned}\widetilde{\Omega}_{ij} &= d\omega_{ij} + \omega_{i0} \wedge \omega_{0j} - \sum_k \omega_{ik} \wedge \omega_{kj}, \qquad (2.3)\\ \widetilde{\Omega}_{\alpha\beta} &= -\frac{1}{2}\, \widetilde{R}_{\alpha\beta\gamma\delta}\omega_\gamma \wedge \omega_\delta,\end{aligned}$$

where $\widetilde{R}_{\alpha\beta\gamma\delta}$ are the components of the curvature tensor of N. The Ricci curvatures are

$$\widetilde{R}_{\alpha\beta} = \widetilde{g}^{\gamma\delta}\widetilde{R}_{\alpha\gamma\beta\delta} = -\widetilde{R}_{\alpha0\beta0} + \sum_j \widetilde{R}_{\alpha j\beta j}. \qquad (2.4)$$

The scalar curvature is

$$\widetilde{R} = \widetilde{g}^{\alpha\beta}\widetilde{R}_{\alpha\beta} = -\widetilde{R}_{00} + \sum_i \widetilde{R}_{ii} = -2\widetilde{R}_{00} + \sum_{i,j} \widetilde{R}_{ijij}. \qquad (2.5)$$

Let M be a spacelike hypersurface in N. We choose a local Lorentzian orthonormal frame field $\{e_\alpha\}$ in N such that, restricted to M, the vectors $\{e_i\}$ are tangent to M, the induced Riemannian metric of M is $g = \sum_i \omega_i^2$ and the induced structural equations of M are

$$\begin{aligned}d\omega_i &= \sum_k \omega_{ik} \wedge \omega_k, \qquad \omega_{ij} + \omega_{ji} = 0,\\ d\omega_{ij} &= -\omega_{i0} \wedge \omega_{0j} + \sum_k \omega_{ik} \wedge \omega_{kj} + \widetilde{\Omega}_{ij}, \qquad (2.6)\\ \Omega_{ij} &= d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2}\, R_{ijkl}\omega_k \wedge \omega_l,\end{aligned}$$

where Ω_{ij}, R_{ijkl} denote the curvature forms and the components of curvature tensor of M respectively.

By Cartan's lemma, we have $\omega_{i0} = h_{ij}\omega_j$, where h_{ij} are components of the second fundamental form of M in N. Then the above structural equations give the Gauss equations

$$R_{ijkl} = \widetilde{R}_{ijkl} - (h_{ik}h_{jl} - h_{il}h_{jk}), \qquad (2.7)$$

and the Codazzi equations

$$\widetilde{R}_{0ijk} = h_{ij,k} - h_{ik,j}. \qquad (2.8)$$

If N satisfies the Einstein equations (1.1), then (2.7), (2.8) give that

$$\begin{aligned}T_{00} &= \widetilde{R}_{00} - \frac{\widetilde{R}}{2}\, \widetilde{g}_{00}\\ &= \widetilde{R}_{00} + \frac{1}{2}\,(-2\widetilde{R}_{00} + \sum_{i,j} \widetilde{R}_{ijij}) \qquad (2.9)\\ &= \frac{1}{2}\,(R - \sum_{i,j} h_{ij}^2 + H^2),\\ T_{0i} &= \widetilde{R}_{0i} = \sum_j h_{ji,j} - H_{,i}, \qquad (2.10)\end{aligned}$$

where $H = \sum_i h_{ii}$ is the mean curvature.

Proposition 2.1 *If $T_{\alpha\beta} = 0$, then $\widetilde{R}_{\alpha\beta} = 0$.*

Proof. Denote $T = tr_{\widetilde{g}}(T_{\alpha\beta})$, then

$$\begin{aligned} T &= -\widetilde{R}_{00} - \frac{\widetilde{R}}{2} + \sum_i (\widetilde{R}_{ii} - \frac{\widetilde{R}}{2}) \\ &= -\widetilde{R}_{00} + \sum_i \widetilde{R}_{ii} - \frac{n+1}{2} \widetilde{R} \\ &= \frac{1-n}{2} \widetilde{R}. \end{aligned}$$

Therefore $T_{\alpha\beta} = 0$ gives that $\widetilde{R} = 0$, then $\widetilde{R}_{\alpha\beta} = 0$ follows from the Einstein equations. □

3 $HU(1,1)$ Representation and Spinors

In this section, and henceforth section, we always assume N is a 5-dimensional Lorentzian manifold with Lorentzian metric of signature $(-1,1,1,1,1)$, and M is a spacelike hypersurface in N. We shall define the hypersurface spinors along M. We describe them first at the level of linear algebra and then globally on the manifold M.

Denote H be the field of quaternious. The hyper-unitary group $HU(1,1)$ is defined to be the subgroup of $GL(2,H)$ that fixes the standard H-Hermitian symmetric form

$$(p,q) = \bar{p}_1 \cdot q_1 - \bar{p}_2 \cdot q_2$$

where $p = (p_1,p_2)^t, q = (q_1,q_2)^t \in H^2$. The group $HU(1,1) = Spin^0(4,1)$ is the double covering group of connected Lorentz group $SO(4,1)$, see [Ha], p272. Let V be the fundamental representation of $HU(1,1)$ on H^2. For any $X \in End(V)$, denote X^* the adjoint of X under $HU(1,1)$ Hermitian structure. We note that any $A \in HU(1,1)$ if and only if $AA^* = I$, $A^*A = I$. On $End(V)$, we define the operator

$$RT(X) = Re\{Trace(X)\}. \tag{3.1}$$

Proposition 3.1 *RT is well-defined, i.e., RT is independent on the choice of basis. Moreover, for any $X, Y \in End(V)$,*

$$RT(X^*Y) = RT(YX^*) = RT(XY^*).$$

Proof. Choosing a basis, we can write

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \qquad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Then

$$X^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \bar{x}_{11} & \bar{x}_{21} \\ \bar{x}_{12} & \bar{x}_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \bar{x}_{11} & -\bar{x}_{21} \\ -\bar{x}_{12} & \bar{x}_{22} \end{pmatrix},$$

$$Y^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \bar{y}_{11} & \bar{y}_{21} \\ \bar{y}_{12} & \bar{y}_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \bar{y}_{11} & -\bar{y}_{21} \\ -\bar{y}_{12} & \bar{y}_{22} \end{pmatrix},$$

where x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22} are quaternious numbers. We have $RT(X) = Re(x_{11} + x_{22})$. Changing basis, X changes to $A^{-1}XA$ for some $A \in HU(1,1)$. So, for proving the first part of the proposition, we need only show that $RT(A^{-1}XA) = RT(X)$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A \in HU(1,1)$ gives that

$$|a|^2 - |b|^2 = 1, \quad |d|^2 - |c|^2 = 1, \quad a\bar{c} - b\bar{d} = 0.$$

We note that $\overline{xy} = \bar{y}\bar{x}$, $Re(x) = Re(\bar{x})$, $Re(\bar{x}y) = Re(x\bar{y})$ for any quaternious numbers x, y. Therefore,

$$\begin{aligned} RT(A^{-1}XA) &= RT(A^*XA) \\ &= Re(\bar{a}x_{11}a - \bar{c}x_{21}a + \bar{a}x_{12}c - \bar{c}x_{22}c \\ &\quad -\bar{b}x_{11}b + \bar{d}x_{21}b - \bar{b}x_{12}d + \bar{d}x_{22}d) \\ &= Re(|a|^2\bar{x}_{11} - |c|^2\bar{x}_{22} - |b|^2\bar{x}_{11} + |d|^2\bar{x}_{22} \\ &\quad -c\bar{a}\bar{x}_{21} + d\bar{b}\bar{x}_{21} + a\bar{c}\bar{x}_{12} - b\bar{d}\bar{x}_{12}) \\ &= Re(\bar{x}_{11} + \bar{x}_{22}) \\ &= RT(X). \end{aligned}$$

For the proof of the second part, since

$$\begin{aligned} RT(X^*Y) &= Re(\bar{x}_{11}y_{11} - \bar{x}_{12}y_{12} - \bar{x}_{21}y_{21} + \bar{x}_{22}y_{22}), \\ RT(YX^*) &= Re(y_{11}\bar{x}_{11} - y_{12}\bar{x}_{12} - y_{21}\bar{x}_{21} + y_{22}\bar{y}_{22}), \\ RT(XY^*) &= Re(x_{11}\bar{y}_{11} - x_{12}\bar{y}_{12} - x_{21}\bar{y}_{21} + x_{22}\bar{y}_{22}). \end{aligned}$$

Hence it follows. □

Corollary 3.1 *On $End(V)$, inner product*

$$\langle X,\, Y\rangle = -\frac{1}{2}\, RT(X^*Y) \tag{3.2}$$

is independent on the choice of basis.

Set

$$\aleph = \{X \in End(V) : X = X^*\}. \tag{3.3}$$

It is independent on the choice of basis since $(A^*XA)^* = A^*X^*A = A^*XA$ for any $X \in \aleph$, $A \in HU(1,1)$.

Proposition 3.2 *On $\aleph$, $Trace(X)$ is independent on the choice of basis.*

Proof. Choosing a basis, let X given by a matrix as above. Then, $X = X^*$ gives that $x_{11} = \bar{x}_{11}$, $x_{22} = \bar{x}_{22}$, $x_{12} = -\bar{x}_{21}$. Hence x_{11}, x_{22} are real numbers, and $RT(X) = x_{11} + x_{22} = Trace(X)$. □

Proposition 3.3 *Set*

$$\aleph_0 = \{X \in \aleph, \;\; Trace(X) = 0\} \tag{3.4}$$

then $(\aleph_0, \;\; \|\;\|) = (R^{4,1}, \;\; \tilde{g})$, *where $\tilde{g}$ is the standnd Lorentzian metric on $R^{4,1}$.*

Proof. Choosing a basis, and take $x_{11} = x_0$, $x_{22} = -x_0$, $x_{12} = x_1 + x_2 i + x_3 j + x_4 k \equiv x$, x_0, x_1, x_2, x_3, x_4 are all real numbers. It gives

$$X = X^* = \begin{pmatrix} x_0 & x \\ -\bar{x} & -x_0 \end{pmatrix}$$

for any $X \in \aleph_0$. Obviously, $\|X\|^2 = \langle X, \; X \rangle = -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = \tilde{g}(X, X)$. Hence we can identify any $X = (x_0, x_1, x_2, x_3, x_4) \in R^{4,1}$ as an element in $\aleph_0$ with norm $\|X\|$, under a basis, which is given by the matrix

$$X = \begin{pmatrix} x_0 & x \\ -\bar{x} & -x_0 \end{pmatrix}, \tag{3.5}$$

where $x = x_1 + x_2 i + x_3 j + x_4 k$. Moreover, this identification does not depend on the choice of basis. □

Choosing an orthornormal basis $\{e_\alpha\}$ on $R^{4,1}$ with e_0 timelike, let $\{e^\alpha\}$ be its dual basis. Then we have the following representations on V,

$$\begin{aligned} e^0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ e^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ e^2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ e^3 &= \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \\ e^4 &= \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}. \end{aligned} \tag{3.6}$$

Now we see the Lie algebra $hu(1,1)$ of Lie group $HU(1,1)$. For any $a \in hu(1,1)$,

$$(\exp ta^*)(\exp ta) = (\exp ta)^*(\exp ta) = I.$$

Hence

$$a^* + a = 0, \tag{3.7}$$

where a^* is the adjoint of a under $HU(1,1)$ Hermitian structure.

Let

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then (3.7) gives

$$\begin{pmatrix} \bar{a}_{11} + a_{11} & -\bar{a}_{21} + a_{12} \\ -\bar{a}_{21} + a_{21} & \bar{a}_{22} + a_{22} \end{pmatrix} = 0.$$

Therefore a_{11}, a_{22} are purely imaginary, and $a_{12} = \bar{a}_{21}$. Hence $dim_R hu(1,1) = 10$. In terms of the representations of $\{e^\alpha\}$ given by (3.6), we can obtain

$$hu(1,1) = span_R\{e^\alpha e^\beta, \quad \alpha \neq \beta\}.$$

We note that

$$so(4,1) = span_R\{e^\alpha \wedge e^\beta, \quad \alpha \neq \beta\}.$$

This gives $hu(1,1) \cong so(4,1)$.

The spinors for $Spin^0(4,1) = HU(1,1)$ structure is just defined by V. This space has a $HU(1,1)$ invariant Hermitian inner product defined by

$$(\phi, \psi) = \bar{\xi_1} \cdot \eta_1 - \bar{\xi_2} \cdot \eta_2 \tag{3.8}$$

for $\phi = (\xi_1, \xi_2)^t \in V, \psi = (\eta_1, \eta_2)^t \in V$. This inner product is not positive definite.

We define the Clifford Multiplication map "."

$$\begin{aligned} . \quad &: R^{4,1} \otimes V \longrightarrow V \\ &X.\, \phi = X\phi, \end{aligned}$$

where X is the correspondent element in $\aleph_0$ for point in $R^{4,1}$, choosing a basis , given by the matrix (3.5). Obviously,

$$\begin{aligned} X.\, X. \quad &= \begin{pmatrix} x_0 & x \\ -\bar{x} & -x_0 \end{pmatrix} \begin{pmatrix} x_0 & x \\ -\bar{x} & -x_0 \end{pmatrix} \\ &= -\tilde{g}(X,X) \cdot Id. \end{aligned}$$

By polarization,

$$X.\,Y. \;+ Y.\,X. \;= -2\tilde{g}(X,Y)\cdot Id.$$

So by the universal property of Clifford algebra, the map "." can be extended to a quaternious representation of Clifford algebra $Cl(4,1)$, hence to the group $HU(1,1)$.

The choice of a timelike covector e^0 yields a diagram

$$\begin{array}{ccc} Sp(1)\times Sp(1) & \xrightarrow{\hat{\alpha}} & HU(1,1) \\ \rho_1 \downarrow & & \downarrow \rho_2 \\ SO(4) & \xrightarrow{\alpha} & SO(4,1), \end{array}$$

where the maps are defined as follows: write $x = x_1 + x_2 i + x_3 j + x_4 k \in H \cong R^4$, $X = \begin{pmatrix} x_0 & x \\ -\bar{x} & -x_0 \end{pmatrix} \in \aleph_0 \cong R^{4,1}$, for $p,\, q \in Sp(1)$, $A \in HU(1,1)$, $a \in SO(4)$,

$$\begin{aligned} \rho_1((p,q))x &= px\bar{q}, \\ \rho_2(A)X &= AXA^*, \\ \hat{\alpha}((p,q)) &= \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \\ \alpha(a)X &= \begin{pmatrix} x_0 & ax \\ -\overline{ax} & -x_0 \end{pmatrix}. \end{aligned}$$

The double-covering map ρ_1 is well-known and we refer to [Sa]. Since

$$\begin{aligned} \langle \rho_2(X),\, \rho_2(Y)\rangle &= \langle AXA^*,\, AYA^*\rangle \\ &= -\frac{1}{2}\, RT((AXA^*)^* AYA^*) \\ &= -\frac{1}{2}\, RT(AXYA^*) \\ &= -\frac{1}{2}\, RT((AX)^* AY) \\ &= -\frac{1}{2}\, RT(X^* Y) \\ &= \langle X,\, Y\rangle, \end{aligned}$$

it implies $\rho_2(A) \in SO(4,1)$. Now we see $Ker(\rho_2)$, for any $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in Ker(\rho_2)$, then $AX = XA$ for any $X \in \aleph \cong R^{4,1}$. Taking $x_0 = 1$, $x = 0$ gives $a_{12} = a_{21} = 0$. Taking $x_0 = 0$, $x = 1$ gives $a_{11} = a_{22}$. Moreover, $a_{11}x = xa_{11}$ for any $x \in H \cong R^4$, then it gives $a_{11} = \pm 1$. Hence $Ker(\rho_2) = Z_2$. This together with $hu(1,1) \cong so(4,1)$ actually imply that ρ_2 is a double-covering map. Also we can obtain in a standand way that $d\rho_2 : hu(1,1) \cong so(4,1)$ given by $d\rho_2(e^\alpha e^\beta) = 2e^\alpha \wedge e^\beta$. Finally,

$$\begin{aligned}
\rho_2 \circ \hat{\alpha}((p,q))X &= \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} \bar{x}_0 & x \\ -\bar{x} & -x_0 \end{pmatrix} \begin{pmatrix} \bar{p} & 0 \\ 0 & \bar{q} \end{pmatrix} \\
&= \begin{pmatrix} px_0\bar{p} & px\bar{q} \\ -q\bar{x}\bar{p} & -qx_0\bar{q} \end{pmatrix} = \begin{pmatrix} x_0 & px\bar{q} \\ -q\bar{x}\bar{p} & -x_0 \end{pmatrix}, \\
\alpha \circ \rho_1((p,q))X &= \begin{pmatrix} x_0 & px\bar{q} \\ -\overline{px\bar{q}} & -x_0 \end{pmatrix} = \begin{pmatrix} x_0 & px\bar{q} \\ -q\bar{x}\bar{p} & -x_0 \end{pmatrix}.
\end{aligned}$$

Therefore

$$\rho_2 \circ \hat{\alpha} = \alpha \circ \rho_1.$$

The above diagram allows us to regard V as $Spin(4) = Sp(1) \times Sp(1)$ representation and gives V an another Hermitian structure. The Clifford multiplication $e^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} :$ $V \to V$ gives an isomorphism $V \cong V$, the new Hermitian structure on V is given by this isomorphism together with the isomorphism $V \cong \overline{V}^*$ given by the $HU(1,1)$ structure. In another word, there is another Hermitian inner product on V given by

$$\begin{aligned}
\langle \phi, \psi \rangle &= (e^0.\phi, \psi) \\
&= ((\xi_1, -\xi_2)^t, (\eta_1, \eta_2)^t) \\
&= \bar{\xi}_1 \cdot \eta_1 - (-\bar{\xi}_2) \cdot \eta_2 \\
&= \bar{\xi}_1 \cdot \eta_1 + \bar{\xi}_2 \cdot \eta_2. \qquad (3.9)
\end{aligned}$$

for $\phi = (\xi_1, \xi_2)^t \in V$, $\psi = (\eta_1, \eta_2)^t \in V$. Hence this new inner product is positive definite and $Sp(1) \times Sp(1)$ invariant.

Proposition 3.4 *For any $X \in R^{4,1}$, spinors $\phi, \psi \in V$, we have*

$$(X.\phi,\ \psi) = (\phi,\ X.\psi).$$

Proof. We note that $X \in \aleph$, thus

$$(X.\phi,\ \psi) = (X\phi,\ \psi) = (X^*\phi,\ \psi) = (\phi,\ X\psi) = (\phi,\ X.\psi).$$

□

Proposition 3.5 *For any $x = (x_1, x_2, x_3, x_4) \in R^4$ regarded as an embedding $X = (0, x_1, x_2, x_3, x_4) \in R^{4,1}$, we have*

$$\langle x.\phi,\ \psi \rangle = -\langle \phi,\ x.\psi \rangle, \qquad \langle e^0.\phi,\ \psi \rangle = \langle \phi,\ e^0.\psi \rangle.$$

Proof.

$$\begin{aligned}\langle x.\phi,\ \psi\rangle &= (e^0.x.\phi,\ \psi) = -(e^0.\phi,\ x.\psi) = -\langle\phi,\ x.\psi\rangle,\\ \langle e^0.\phi,\ \psi\rangle &= (e^0.e^0.\phi,\ \psi) = (e^0.\phi,\ e^0.\psi) = \langle\phi,\ e^0.\psi\rangle.\end{aligned}$$

□

Define the volume form $* = -e^1.e^2.e^3.e^4$.. Direct computation shows

$$* = \begin{pmatrix} -ijk & 0 \\ 0 & ijk \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence $*^2=$ Id. And $V = V^+ \oplus V^-$, where

$$\begin{aligned} V^+ &= \{\phi : *.\phi = \phi\} = \{(\xi,\ 0)^t\},\\ V^- &= \{\phi : *.\phi = -\phi\} = \{(0,\ \eta)^t\}.\end{aligned}$$

Proposition 3.6 *Half spinor spaces V^+, V^- are orthogonal under inner products $(\ ,\ )$ and $\langle\ ,\ \rangle$.*

Proof. Propositions 3.4, 3.5 imply that opertor '$*$' is isometric under these two inner products. For $\phi^+ \in V^+$, $\psi^- \in V^-$, we have

$$\begin{aligned}(\phi^+,\ \psi^-) = (*.\phi^+,\ -*.\psi^-) = -(\phi^+,\ \psi^-),\\ \langle\phi^+,\ \psi^-\rangle = \langle *.\phi^+,\ -*.\psi^-\rangle = -\langle\phi^+,\ \psi^-\rangle.\end{aligned}$$

Therefore

$$(\phi^+,\ \psi^-) = 0, \qquad \langle\phi^+,\ \psi^-\rangle = 0.$$

□

In terms of (3.6), we obtain

Proposition 3.7 *For any $\phi^+ \in V^+$, $\psi^- \in V^-$, we have*

$$e^0.\ \phi^+ = \phi^+,\ \ e^0.\ \psi^- = -\psi^-,\ \ e^i.\ \phi^+ \in V^-,\ \ e^i.\ \psi^- \in V^+.$$

In R^4, space of 2-forms $\bigwedge$ splits as self-dual part $\bigwedge^+$ and anti-self-dual part $\bigwedge^-$ by the Hodge star operator. Where

$$\begin{aligned}\bigwedge^+ &= span\{e^1\wedge e^2 + e^3\wedge e^4, e^1\wedge e^3 + e^4\wedge e^2, e^2\wedge e^3 + e^4\wedge e^1\},\\ \bigwedge^- &= span\{e^1\wedge e^2 - e^3\wedge e^4, e^1\wedge e^3 - e^4\wedge e^2, e^2\wedge e^3 - e^4\wedge e^1\}.\end{aligned}$$

Define the Clifford multiplication of 2-form on V by:

$$(e^i\wedge e^i).\ \phi = e^i.\ e^j.\ \phi, \quad for\ \ i\neq j. \tag{3.10}$$

Proposition 3.8

$$\bigwedge^{+}. V^{-} = 0, \ \bigwedge^{-}. V^{+} = 0.$$

Proof. It shows by (3.6), (3.10) that

$$\begin{aligned}
(e^1 \wedge e^2 + e^3 \wedge e^4).\ \phi &= \begin{pmatrix} 2i & 0 \\ 0 & 0 \end{pmatrix} \phi, \\
(e^1 \wedge e^3 + e^4 \wedge e^2).\ \phi &= \begin{pmatrix} 2j & 0 \\ 0 & 0 \end{pmatrix} \phi, \\
(e^2 \wedge e^3 + e^4 \wedge e^1).\ \phi &= \begin{pmatrix} 2k & 0 \\ 0 & 0 \end{pmatrix} \phi, \\
(e^1 \wedge e^2 - e^3 \wedge e^4).\ \phi &= \begin{pmatrix} 0 & 0 \\ 0 & -2i \end{pmatrix} \phi, \\
(e^1 \wedge e^3 - e^4 \wedge e^2).\ \phi &= \begin{pmatrix} 0 & 0 \\ 0 & -2j \end{pmatrix} \phi, \\
(e^2 \wedge e^3 - e^4 \wedge e^1).\ \phi &= \begin{pmatrix} 0 & 0 \\ 0 & -2k \end{pmatrix} \phi.
\end{aligned} \tag{3.11}$$

Hence the proposition follows. □

From now on, we always assume M is a spin spacelike hypersurface in N. Then, the above algebra facts carry over to vector bundles once a spin structure is choosen. Let $F(N)$ denote the $SO(4,1)$ frame bundle of the cotangent bundle of N and let $i : M \longrightarrow N$ be the inclusion. The required spin structure is a lift of the bundle $i^*F(N)$ to a $HU(1,1)$ bundle over M. But

$$i^*F(N) = F(M) \times_\alpha SO(4,1),$$

so we need only lift the $SO(4)$ frame bundle of M to a $Sp(1) \times Sp(1)$ bundle $\widetilde{F(M)}$. The obstruction to such an $\widetilde{F(M)}$ is the Stiefel-Whitney class $\omega_2(M)$.

Since M is spin, $\omega_2(M) = 0$, $\widetilde{F(M)}$ exists. The number of such lifts $\widetilde{F(M)}$ is then classified by $H^1(M, Z_2)$. Choosing one, we obtain the desired $HU(1,1)$ bundle

$$i^*\widetilde{F(N)} = \widetilde{F(M)} \times_{\hat{\alpha}} HU(1,1)$$

over M and the associated spin vector bundle

$$i^*\widetilde{F(N)} \times_\rho V = \widetilde{F(M)} \times_{\bar{\rho}} V,$$

where ρ is the representation V of $HU(1,1)$, and $\bar{\rho}$ is its restriction to $Sp(1) \times Sp(1)$. This vector bundle – denoted S – carries the inner products $(\ ,\ )$ and $\langle\ ,\ \rangle$. Sections of S

are called Hypersurface Spinors along M. Proposition 3.3 implies $T^*M \cong \aleph_0(S)$, so the Clifford multiplication is globally-defined on M.

The metric connection $\tilde{\nabla}$ on $F(N)$ determines connections on $i^*\widetilde{F(N)}$ and its associated bundle; The resulting connection (also denoted) $\tilde{\nabla}$ on S is compatible with the metric $(\ , \ )$ but not compatible with the metric $\langle \ , \ \rangle$. Let ∇ be the Riemannian connection on $F(M)$. It also induces a connection ∇ on $S = \widetilde{F(M)} \times_{\tilde{\rho}} V$. We shall show that ∇ is compatible with $\langle \ , \ \rangle$.

Fix a point $p \in M$ and an orthonormal basis $\{e_\alpha\}$ of T_pN with e_0 normal and $e_1, \ e_2, \ e_3, \ e_4$ tangent to M. Extend $e_1, \ e_2, \ e_3, \ e_4$ to an orthonormal frame in a neighbourhood of p in M such that

$$(\nabla_i e_j)_p = 0, \qquad 1 \leq i, j \leq 4.$$

Extend this to a local orthonormal $\{e_\alpha\}$ for N with

$$(\tilde{\nabla}_0 e_j)_p = 0, \qquad 1 \leq i \leq 4.$$

Let $\{e^\alpha\}$ be the dual coframe. Then

$$\begin{aligned} (\tilde{\nabla}_i e^j)_p &= -h_{ij} e^0, \\ (\tilde{\nabla}_i e^0)_p &= -h_{ij} e^j \qquad 1 \leq i, j \leq 4, \end{aligned}$$

where $h_{ij} = \langle \tilde{\nabla}_i e_0, \ e_j \rangle$ are the components of the second fundamental form at p.

The connection forms for the metric connection on $F(N)$, $F(M)$ are given by

$$\begin{aligned} \omega^N &= \omega_{\alpha\beta} e^\alpha \wedge e^\beta, \\ \omega^M &= \omega_{ij} e^i \wedge e^j \end{aligned}$$

respectively. The connection forms for induced connections $\tilde{\nabla}$ and ∇ on $i^*\widetilde{F(N)}, \widetilde{F(M)}$ are

$$\begin{aligned} \tilde{\omega}^N &= \frac{1}{2} \, \omega_{\alpha\beta} e^\alpha . e^\beta . \, , \\ \tilde{\omega}^M &= \frac{1}{2} \, \omega_{ij} e^i . e^j . \end{aligned}$$

respectively by the lie algebra isomorphism $\rho : so(4,1) \cong hu(1,1)$, $\rho(e^\alpha \wedge e^\beta) = \frac{1}{2} \, e^\alpha . e^\beta . \,$. Since $\omega_{0i} = h_{ij}\omega^j$ along M, we have the following relations about connections on S,

$$\tilde{\nabla} = \nabla + \frac{1}{2} \, h_{jk} \omega^k \otimes e^0 . e^j . \, , \tag{3.12}$$

$$\tilde{\nabla}_i = \nabla_i + \frac{1}{2} \, h_{ji} e^0 . e^j . \, . \tag{3.13}$$

Proposition 3.9 *The induced connection ∇ on S is compatible with the metric $\langle\ ,\ \rangle$.*

Proof. In the above frame we have, at $p \in M$, $(\widetilde{\nabla}_i e^0)_p = -h_{ij}e^j$, then

$$\begin{aligned} d(\langle\phi,\ \psi\rangle * e_i) &= d((e^0.\phi,\ \psi) * e_i) \\ &= ((-h_{ij}e^j.\phi,\ \psi) + (e^0.\widetilde{\nabla}_i\phi,\ \psi) + (e^0.\phi,\ \widetilde{\nabla}_i\psi)) * 1 \\ &= ((-h_{ij}e^j.\phi,\ \psi) + (e^0.\nabla_i\phi,\ \psi) + (e^0.\phi,\ \nabla_i\psi) + \\ &\quad (e^0.\frac{1}{2}\, h_{ij}e^0.e^j.\phi,\ \psi) + (e^0.\phi,\ \frac{1}{2}\, h_{ij}e^0.e^j.\psi)) * 1 \\ &= (\langle\nabla_i\phi,\ \psi\rangle + \langle\phi,\ \nabla_i\psi\rangle) * 1. \end{aligned}$$

□

Finally, we define a corresponding set of "constant spinors" for each end M_l. Let Θ : $R^4 - K_l \longrightarrow M_l$ be the diffeomorphism which defines M_l. The pullback bundle $\Theta_l^*\widetilde{F(M)}$ differs from the trivial spin bundle over $R^4 - K_l$ by an element of $H^1(R^4 - K_l;\ Z_2) = 0$. Hence the spin structure is trivial over the end M_l and the bundle $\Theta_l^* S$ extends trivially over all of R^4. The $\Theta_l^{-1}-$ pullbacks of the constant sections of the bundle $R^4 \times S$ over R^4 then provide a set of "constant spinors" over the M_l.

4 The Hypersurface Dirac Operators

The hypersurface spinor bundle S along M splits as a direct sum of positive half spinor bundle S^+ and negative half spinor bundle S^-. The connection ∇ preserves this splittings since ∇ commutes with operator $* = -e^1.e^2.e^3.e^4.$, but the connection $\widetilde{\nabla}$ does not preserve this splittings.

Denote c the Clifford multiplication ".", the usually Dirac operator on M defined by ∇ on S is the composition

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{c} \Gamma(S).$$

The hypersurface Dirac operator – denoted $\widetilde{D}$ – is defined by the second connection $\widetilde{\nabla}$ on S. Intrinsically, $\widetilde{D}$ is the composition

$$\Gamma(S) \xrightarrow{\widetilde{\nabla}} \Gamma(T^*M \otimes S) \xrightarrow{c} \Gamma(S).$$

In a local orthonormal coframe $\{e^i\}$ of M,

$$D\phi = e^i.\nabla_i\phi, \qquad \widetilde{D}\phi = e^i.\widetilde{\nabla}_i\phi$$

for any $\phi \in \Gamma(S)$.

By Propositions 3.5, 3.9, D is self-adjoint under the metric $\langle\ ,\ \rangle$. Also we have the classical Lichnerowicz formula:

$$D^*D = D^2 = \nabla^*\nabla + \frac{R}{4}, \tag{4.1}$$

where R is the scalar curvature of M.

Lemma 4.1 *For any $\phi \in \Gamma(S)$, we have*

$$\widetilde{D}\phi = D\phi + \frac{H}{2} e^0.\phi, \tag{4.2}$$

where $H = \sum h_{ii}$ is the mean curvature.

Proof. Since $h_{ij} = h_{ji}$, and $e^i.e^j. = -e^j.e^i.$, for $i \neq j$, then (3.13) gives

$$\widetilde{D}\phi = e^i.\widetilde{\nabla}_i\phi = e^i.\nabla_i\phi + \frac{1}{2}\, h_{ij}e^i.e^0.e^j.\phi = D\phi + \frac{H}{2}\, e^0.\phi.$$

□

We note that writing $\phi = \phi^+ + \psi^-$ where $\phi^+ \in S^+$, $\psi^- \in S^-$, we have $e^0.\phi^+ = \phi^+$, $e^0.\psi^- = -\psi^-$.

Lemma 4.2

$$\begin{aligned} d(\langle e^i.\phi,\ \psi\rangle * e^i) &= (\langle D\phi,\ \psi\rangle - \langle \phi,\ D\psi\rangle) * 1 = (\langle \widetilde{D}\phi,\ \psi\rangle - \langle \phi,\ \widetilde{D}\psi\rangle) * 1, \\ d(\langle \phi,\ \widetilde{\nabla}_i\psi\rangle * e^i) &= (\langle \widetilde{\nabla}_i\phi,\ \widetilde{\nabla}_i\psi\rangle - \langle \phi,\ (-\widetilde{\nabla}_i + h_{ij}e^0.e^j.)\widetilde{\nabla}_i\psi\rangle) * 1. \end{aligned}$$

Proof.

$$\begin{aligned} d(\langle e^i.\phi,\ \psi\rangle * e^i) &= (\langle e^i.\nabla_i\phi,\ \psi\rangle - \langle e^i.\phi,\ \nabla_i\psi\rangle) * 1 \\ &= (\langle D\phi,\ \psi\rangle - \langle \phi,\ D\psi\rangle) * 1 \\ &= (\langle \widetilde{D}\phi - \frac{H}{2}\, e^0.\phi,\ \psi\rangle - \langle \phi,\ \widetilde{D}\psi - \frac{H}{2}\, e^0.\psi\rangle) * 1 \\ &= (\langle \widetilde{D}\phi,\ \psi\rangle - \langle \phi,\ \widetilde{D}\psi\rangle) * 1. \\ d(\langle \phi,\ \widetilde{\nabla}_i\psi\rangle * e^i) &= (\langle \nabla_i\phi,\ \widetilde{\nabla}_i\psi\rangle - \langle \phi,\ \nabla_i\widetilde{\nabla}_i\psi\rangle) * 1 \\ &= (\langle (\widetilde{\nabla}_i\phi - \frac{1}{2}\, h_{ij}e^0.e^j.\phi),\ \widetilde{\nabla}_i\psi\rangle - \\ &\quad \langle \phi,\ (\widetilde{\nabla}_i - \frac{1}{2}\, h_{ij}e^0.e^j.\psi)\widetilde{\nabla}_i\psi\rangle) * 1 \\ &= (\langle \widetilde{\nabla}_i\phi,\ \widetilde{\nabla}_i\psi\rangle - \langle \phi,\ (-\widetilde{\nabla}_i + h_{ij}e^0.e^j.)\widetilde{\nabla}_i\psi\rangle) * 1. \end{aligned}$$

□

Corollary 4.1 $D^* = D, \quad \widetilde{D}^* = \widetilde{D}, \quad \widetilde{\nabla}_i^* = -\widetilde{\nabla}_i + h_{ij}e^0.e^j..$

Now we derive the following two Weitzenböck formulas, the second was given by Witten [W1, P-T]. Our approach is a little different from them.

Theorem 4.1 *For any $\phi \in \Gamma(S)$,*

$$\begin{aligned}\widetilde{D}^2\phi &= \nabla^*\nabla\phi + \frac{1}{4}\,(R+H^2)\phi - \frac{1}{2}\,\nabla_i H e^0.e^i.\phi &(4.3)\\ &= \widetilde{\nabla}^*\widetilde{\nabla}\phi + \frac{1}{2}\,(T_{00}+T_{0i}e^0.e^i.)\phi. &(4.4)\end{aligned}$$

Proof. Since

$$\begin{aligned}\nabla_i(e^0.\phi) &= (\widetilde{\nabla}_i - \frac{1}{2}\,h_{ij}e^0.e^j.)(e^0.\phi)\\ &= -h_{ij}e^j.\phi + e^0.\widetilde{\nabla}_i\phi + \frac{1}{2}\,h_{ij}e^j.\phi\\ &= e^0.(\widetilde{\nabla}_i - \frac{1}{2}\,h_{ij}e^0.e^j.)\phi = e^0.\nabla_i\phi,\end{aligned}$$

then the Lemma 4.1 and Lichnerowicz formula (4.1) show

$$\begin{aligned}\widetilde{D}^2\phi &= (D + \frac{H}{2}\,e^0.)(D\phi + \frac{H}{2}\,e^0.\phi)\\ &= D^2\phi + \frac{H^2}{4}\,e^0.e^0.\phi + \frac{1}{2}\,e^i.\nabla_i H e^0.\phi\\ &= \nabla^*\nabla\phi + \frac{R}{4}\,\phi + \frac{H^2}{4}\,\phi - \frac{1}{2}\,\nabla_i H e^0.e^i.\phi.\end{aligned}$$

But

$$\begin{aligned}\widetilde{\nabla}^*\widetilde{\nabla}\phi &= (-\widetilde{\nabla}_i + h_{ij}e^0.e^j.)\widetilde{\nabla}_i\phi\\ &= (-\nabla_i + \frac{1}{2}\,h_{ij}e^0.e^j.)(\nabla_i\phi + \frac{1}{2}\,h_{ik}e^0.e^k.\phi)\\ &= \nabla^*\nabla\phi - \frac{1}{4}h_{ij}h_{ik}e^j.e^k.\phi - \frac{1}{2}\nabla_i(h_{ij}e^0.e^j.\phi) + \frac{1}{2}h_{ij}e^0.e^j.\nabla_i\phi\\ &= \nabla^*\nabla\phi + \frac{1}{4}\,\sum_{i,j} h_{ij}^2\phi - \frac{1}{2}\,\nabla_i h_{ij}e^0.e^j.\phi.\end{aligned}$$

Substituting it into (4.4) and using (2.9), (2.10), we obtain,

$$\begin{aligned}\widetilde{D}^2\phi &= \widetilde{\nabla}^*\widetilde{\nabla}\phi + \frac{1}{4}\,(R - |A|^2 + H^2)\phi + \frac{1}{2}\,(\nabla_j h_{ji} - \nabla_i H)e^0.e^i.\phi\\ &= \widetilde{\nabla}^*\widetilde{\nabla}\phi + \frac{1}{2}\,(T_{00}+T_{0i}e^0.e^i.)\phi.\end{aligned}$$

□

In terms of Lemma 4.2, we get the integral form of the Weitzenböck formula.

$$\int_M |\widetilde{\nabla}\phi|^2 + \langle\phi,\ \widetilde{R}\phi\rangle - |\widetilde{D}\phi|^2 = \frac{1}{2}\,\int_{\partial M}\langle\phi,\ [e^i,\ e^j].\widetilde{\nabla}_j\phi\rangle * e^i, \tag{4.5}$$

where $\widetilde{R} = \frac{1}{2}\,(T_{00}+T_{0i}e^0.e^i.)$, and $[e^i,\ e^j]. = e^i.e^j. - e^j.e^i..$

5 Boundary Value Problems, Positive Mass Conjecture

In this section, we assume M is a spin spacelike asymptotically flat hypersurface of order $\tau > 1$. We shall study the infinity boundary value problems for the hypersurface Dirac equation. We simplify the original arguments in [P-T]. Finally, we prove the Positive Mass Conjecture I in our case.

Lemma 5.1 *Suppose that ϕ, $\{\phi_i\}$ are C^1 hypersurface spinors along M and satisfy*

$$\widetilde{\nabla}\phi = 0, \quad \widetilde{\nabla}\phi_i = 0 \quad for \ each \ i$$

(i) If $\lim_{x\to\infty}\phi(x) = 0$, where the limit is taken along M in one asymptotic end, then $\phi = 0$.
(ii) If $\{\phi_i\}$ are linearly independent in some end, then they are linearly independent everywhere on M.

Proof. (i) The assumption $\widetilde{\nabla}\phi = 0$ gives that $\nabla_i\phi = -\frac{1}{2}h_{ij}e^0.e^j.\phi$. Therefore

$$2|\phi||d|\phi|| = |d|\phi|^2| = |[\langle\nabla_i\phi, \phi\rangle + \langle\phi, \nabla_i\phi\rangle]e^i| \leq |h||\phi|^2.$$

On each end, since $h = O(r^{-\tau-1})$, this gives

$$|d|\phi|| \leq Cr^{-\tau-1}$$

on the complement of the zero set of ϕ. Integrating this along a path from $x_0 \in M$ gives

$$|\phi(x)| \geq |\phi(x_0)|e^{C(|x_0|^{-\tau}-|x|^{-\tau})}.$$

Taking x to be the first zero of ϕ along the path of integration, or taking the limit as $|x| \to \infty$ if no such zero exists, shows that $\phi(x_0) = 0$. Hence $\phi = 0$ on the ends. On the compact set K, since h is bounded, we have

$$|\phi(x)| \geq |\phi(x_0)|e^{C(|x_0|-|x|)}.$$

Hence $\phi = 0$ on K in taking the path to the ends.

(ii) It follows from the first part. □

Remark 5.1 *ϕ may not be zero if the decay of h is not faster than $O(r^{-1})$.*

By (1.2), (1.3) (1.4) and Lemma 4.1, $\widetilde{D}$ gives the maps for the following weighted Hölder spaces

$$C^{2,\alpha}_{-\tau}(S) \xrightarrow{\widetilde{D}} C^{1,\alpha}_{-\tau-1}(S) \xrightarrow{\widetilde{D}} C^{0,\alpha}_{-\tau-2}(S) \tag{5.1}$$

Lemma 5.2 *On $C^{2,\alpha}_{-\tau}(S)$, we have, for the maps (5.1),*

$$Ker(\widetilde{D}) = Ker(\widetilde{D}^2).$$

Proof. Obviously, $Ker(\widetilde{D}) \subset Ker(\widetilde{D}^2)$. Let $\phi \in C^{2,\alpha}_{-\tau}(S)$ such that $\widetilde{D}^2\phi = 0$. Then by Lemma 4.2

$$d(\langle e^i.\phi,\ \widetilde{D}\phi\rangle * e^i) = (\langle\phi,\ \widetilde{D}^2\phi\rangle - \langle\widetilde{D}\phi,\ \widetilde{D}\phi\rangle) * 1 = -|\widetilde{D}\phi|^2 * 1.$$

Thus

$$-\int_M |\widetilde{D}\phi|^2 * 1 = \int_{\partial M} \langle e^i.\phi,\ \widetilde{D}\phi\rangle * e^i.$$

But $\langle e^i.\phi,\ \widetilde{D}\phi\rangle = O(r^{-2\tau-1})$, and $Vol(\partial M) = O(r^{-3})$ by (1.2), (1.3). Hence the right side of the above integral vanishes. Therefore $\widetilde{D}\phi = 0$ on M. This gives $Ker(\widetilde{D}^2) \subset Ker(\widetilde{D})$ and we complete the proof. □

Lemma 5.3 *If the dominant energy condition holds on M, then the maps (5.1) is injective for each $\widetilde{D}$.*

Proof. Direct computation shows

$$T_{00} + T_{0i}e^0.e^i. = \begin{pmatrix} T_{00} & T_0 \\ \overline{T}_0 & T_{00} \end{pmatrix},$$

where $T_0 = T_{01} + T_{02}i + T_{03}j + T_{04}k$. It is semi-positive definite when M satisfies the dominant energy condition. Thus $\langle\phi, \widetilde{R}\phi\rangle \geq 0$. If $\phi \in Ker(\widetilde{D})$ for either $\widetilde{D}$, $\phi \in C^{2,\alpha}_{-\tau}(S)$ or $\phi \in C^{1,\alpha}_{-\tau-1}(S)$, then $\lim_{r\to\infty}\phi = 0$. Furthermore,

$$\langle\phi,\ [e^i,\ e^j].\widetilde{\nabla}_j\phi\rangle = \langle\phi,\ [e^i,\ e^j].(\nabla_j\phi + \frac{1}{2}\ h_{ij}e^0.e^j.\phi)\rangle = O(r^{-2\tau-1}).$$

Hence (4.5) gives $\widetilde{\nabla}\phi = 0$ on M and we complete the proof by Lemma 5.1 (i). □

We recall the following theorem about the weighted elliptic regularity. For the proof, we refer to [L-P].

Theorem 5.1 *If $0 < \beta < 2$, $h \in C^{0,\alpha}_{-\delta}(S)$ for some $\delta > 2$, and the operator*

$$\nabla^*\nabla + h\ :\ C^{2,\alpha}_{-\beta}(S) \longrightarrow C^{0,\alpha}_{-\beta-2}(S)$$

is injective, then it is isomorphism.

Lemma 5.4 *If the dominant energy condition holds on M, then the map*

$$\widetilde{D}^2\ :C^{2,\alpha}_{-\tau}(S) \longrightarrow C^{0,\alpha}_{-\tau-2}(S)$$

is an isomorphism.

Proof. (1.2) (1.3) and (1.4) show that

$$(\frac{1}{4}\,(R+H^2)-\frac{1}{2}\,\nabla_i He^0.e^i.\,)\in C^{0,\alpha}_{-\tau-1}(S)$$

Hence (4.4) and the above theorem give the proof. □

Lemma 5.5 *If the dominant energy condition holds on M, then the maps (5.1) is an isomorphism for each $\widetilde{D}$.*

Proof. Each $\widetilde{D}$ is injective but $\widetilde{D}^2$ is surjective. Hence each $\widetilde{D}$ is surjective. □

Theorem 5.2 *If the dominant energy condition holds on M, then for any constant spinor ϕ_0 on ends, the following boundary value problem has a unique solution $\phi \in C^{2,\alpha}(S)$.*

$$\left\{\begin{array}{rcl} \widetilde{D}\phi & = & 0 \\ \lim_{r\to\infty}\phi & = & \phi_0. \end{array}\right. \tag{5.2}$$

Proof. The hypothesis on asymptotical metric, on each end, allow for an orthonormal coframe $\{e^i\}$ with $|e^i - dx^i| = O(r^{-\tau-1})$, where $\{dx^i\}$ is the asympotical coordinates on the end. In fact, orthonormalizing $\{dx^i\}$ yields an orthonormal coframe

$$e^i = dx^i + \frac{1}{2}\,a_{ik}dx^k + O(r^{-\tau-1}) \tag{5.3}$$

which gives the required coframe $\{e^i\}$. And

$$\nabla_j = \partial_j - \frac{1}{4}\,\Gamma_{kjl}dx^k.dx^l. + O(r^{-\tau-1}),$$

where

$$\Gamma_{kjl} = \frac{1}{2}\,(\partial_j g_{kl} + \partial_l g_{kj} - \partial_k g_{jl}) = O(r^{-\tau-1}).$$

For constant spinor ϕ_0, $\partial_j\phi_0 = 0$, we have

$$\begin{aligned} \widetilde{D}\phi_0 &= e^i.\nabla_i\phi_0 + \frac{H}{2}\,e^0.\phi_0 \\ &= -\frac{1}{4}\,\Gamma_{kjl}dx^j.dx^k.dx^l.\phi_0 + \frac{H}{2}\,e^0.\phi_0 + O(r^{-\tau-1}). \end{aligned}$$

Hence $\widetilde{D}\phi_0 \in C^{1,\alpha}_{-\tau-1}(S)$. By Lemma 5.5, there is unique $\phi_1 \in C^{2,\alpha}_{-\tau}(S)$ such that $\widetilde{D}\phi_1 = -\widetilde{D}\phi_0$. Take $\phi = \phi_1 + \phi_0$. Obviously, ϕ is the unique solution of (5.2). □

Positive Mass Conjecture *Let N be a 5-dimensional Lorentzian manifold with Lorentzian metric $\tilde{g}$ of signature $(-1,1,1,1,1)$, $M \subset N$ be a spin spacelike asymptotically flat hypersurface of order $\tau > 1$. If the dominant energy condition holds on M, then, on each end M_l,*

$$E_l \geq |P_l| \equiv (\sum_{k=1}^{4} p_{lk}^2)^{\frac{1}{2}}.$$

If $E_{l_0} = 0$ for some l_0, then M has only one end and N is flat along M.

Proof. For end M_l, let constant spinor ϕ_0 be $|\phi_0| = 1$ on M_l, and $\phi_0 = 0$ on other ends. Denote $\phi = \phi_1 + \phi_0$, where $\phi_1 \in C^{2,\alpha}_{-\tau}(S)$ be the corresponding solution of (5.2) for this ϕ_0. Thus, (4.5) gives, under the coframe $\{e^i\}$ choosen in (5.3),

$$\begin{aligned} 2\int_M |\widetilde{\nabla}\phi|^2 + \langle \phi, \widetilde{R}\phi\rangle &= \int_{\partial M} \langle \phi, [e^i, e^j].\widetilde{\nabla}_j\phi\rangle * e^i \\ &= \int_{\partial M} \langle \phi_0, [e^i, e^j].\widetilde{\nabla}_j\phi_0\rangle * e^i + \sum, \end{aligned}$$

where

$$\begin{aligned} \sum &= \int_{\partial M} (\langle \phi_1, [e^i, e^j].\widetilde{\nabla}_j\phi_0\rangle + \langle \phi_0, [e^i, e^j].\widetilde{\nabla}_j\phi_1\rangle + \langle \phi_1, [e^i, e^j].\widetilde{\nabla}_j\phi_1\rangle) * e^i \\ &= \int_{\partial M} \langle \phi_1, [e^i, e^j].\nabla_j\phi_0\rangle * e^i + \int_{\partial M} \langle \phi_1, \frac{1}{2}\, h_{jk}[e^i, e^j].e^0.e^k.\phi_0\rangle * e^i + \\ &\quad \int_{\partial M} \langle -\nabla_j\phi_0, [e^i, e^j].\phi_1\rangle * e^i + \int_{\partial M} d\langle \phi_0, [e^i, e^j].\phi_1\rangle * (e^i \wedge e^j) + \\ &\quad \int_{\partial M} \langle \phi_0, \frac{1}{2}\, h_{jk}[e^i, e^j].e^0.e^k.\phi_1\rangle * e^i + \int_{\partial M} \langle \phi_1, [e^i, e^j].\widetilde{\nabla}_j\phi_1\rangle * e^i \\ &= 2Re\int_{\partial M} \langle \phi_1, [e^i, e^j].\nabla_j\phi_0\rangle * e^i + \int_{\partial M} \langle \phi_1, [e^i, e^j].\nabla_j\phi_1\rangle * e^i + \\ &\quad Re\int_{\partial M} h_{jk}\langle \phi_1, [e^i, e^j].e^0.e^k.\phi_0\rangle * e^i + \frac{1}{2}\int_{\partial M} h_{jk}\langle \phi_1, [e^i, e^j].e^0.e^k.\phi_1\rangle * e^i. \end{aligned}$$

Since $\phi_1 = O(r^{-\tau})$, $\nabla_j\phi_0 = O(r^{-\tau-1})$, $h_{ij} = O(r^{-\tau-1})$, then $\sum = 0$. Therefore

$$\begin{aligned} 2\int_M |\widetilde{\nabla}\phi|^2 + \langle \phi, \widetilde{R}\phi\rangle &= \int_{\partial M} (\langle \phi_0, [dx^i, dx^j].\widetilde{\nabla}_j\phi_0\rangle + O(r^{-2\tau-1})) * d\Omega^i \\ &= \int_{\partial M} \langle \phi_0, -\frac{1}{8}\,\Gamma_{kjl}[dx^i, dx^j].[dx^k, dx^l].\phi_0\rangle * d\Omega^i + \\ &\quad \int_{\partial M} \langle \phi_0, \frac{1}{2}\, h_{jk}[dx^i, dx^j].dx^0.dx^k.\phi_0\rangle * d\Omega^i \\ &= \int_{\partial M} \frac{1}{2}\langle \phi_0, \Gamma_{kjl}(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il})\phi_0\rangle * d\Omega^i + \\ &\quad \int_{\partial M} \langle \phi_0, h_{jk}(\delta^{ij} + dx^i.dx^j.)dx^0.dx^k.\phi_0\rangle * d\Omega^i. \end{aligned}$$

Since $\Gamma_{kjl} = \Gamma_{klj}$, we have

$$\begin{aligned} \Gamma_{kjl}(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il}) &= \sum_{j,l,j\neq i} \Gamma_{ijl}\delta^{jl} - \sum_{j,l,k\neq i,j\neq l} \Gamma_{kjl}\delta^{il}\delta^{jk} \\ &= \Gamma_{ijj} - \Gamma_{jji} \\ &= \frac{1}{2}\,(\partial_j g_{ij} + \partial_j g_{ij} - \partial_i g_{jj} - \partial_j g_{ji} - \partial_i g_{jj} + \partial_j g_{ji}) \\ &= \partial_j g_{ij} - \partial_i g_{jj}, \\ h_{jk}(\delta^{ij} + dx^i.dx^j).dx^0.dx^k. &= h_{ik}dx^0.dx^k. + h_{jj}dx^i.dx^j.dx^0.dx^j. \\ &= (h_{ik} - \delta_{ik}h_{jj})dx^0.dx^k. \\ &= (h_{ik} - \delta_{ik}h_{jj})e^0.e^k. + O(r^{-2\tau-1}). \end{aligned}$$

Therefore

$$2\int_M |\widetilde{\nabla}\phi|^2 + \langle \phi,\ \widetilde{R}\phi\rangle = \frac{1}{2}\left(\langle \phi_0,\ E_l\phi_0\rangle + \langle \phi_0,\ p_{lk}e^0.e^k.\phi_0\rangle\right).$$

But

$$p_{lk}e^0.e^k. = \begin{pmatrix} 0 & p_l \\ \bar{p}_l & 0 \end{pmatrix},$$

where $p_l = p_{l1} + p_{l2}i + p_{l3}j + p_{l4}k$ has real eigenvalues $\lambda = \pm|P_l|$. Now we take ϕ_0 to be the eigenspinor of eigenvalue $-|P_l|$ with $|\phi_0| = 1$. In terms of this constant spinor, we finally obtain

$$E_l - |P_l| = 4\int_M |\widetilde{\nabla}\phi|^2 + \langle \phi,\ \widetilde{R}\phi\rangle \geq 0$$

Thus the proof of the first part is complete.

Now suppose that $E_1 = 0$. Choose a basis $\{\psi_{0c}|c = 1,\ 2\}$ of constant spinors and take as asymptotic data the constant spinors $\{\psi_{lc}|c = 1,\ 2\}$ with $\psi_{1c} = \psi_{0c}$ on M_1 and $\psi_{1c} = 0$ on all other ends M_l. Let ψ_c be the solutions of $\widetilde{D}\psi_c = 0$ constructed from this data by Theorem 5.2. The vanishing of E_1 then implies $\widetilde{\nabla}\psi_c = 0$ and $\psi_c \to 0$ uniformly on each end except M_1. But this contradicts Lemma 5.1 (i) unless M_1 is the only end of M.

Because $\{\psi_c\}$ are linearly independent on M_1 they are linearly independent everywhere by Lemma 5.1 (ii). Furthermore, $\widetilde{\nabla}\psi_c = 0$, so in a local frame $\{e_i\}$ of M,

$$0 = (\widetilde{\nabla}_i\widetilde{\nabla}_j - \widetilde{\nabla}_j\widetilde{\nabla}_i - \widetilde{\nabla}_{[e_i,\ e_j]})\psi_c = -\frac{1}{4}\,\widetilde{R}_{\alpha\beta ij}e^\alpha.e^\beta.\psi_c$$

for all $1 \leq i,\ j \leq 4$. This implies

$$\widetilde{R}_{\alpha\beta ij} = 0, \quad 1 \leq i,\ j \leq 4. \tag{5.4}$$

Then the Einstein's equations give

$$T_{00} = \frac{1}{2}\sum \widetilde{R}_{ijij} = 0.$$

And the dominant energy condition shows

$$|T_{\alpha\beta}| \leq |T_{00}| = 0.$$

Hence Proposition 2.1 gives $\widetilde{R}_{\alpha\beta} = 0$. This together with (5.4) implies

$$\widetilde{R}_{\alpha\beta\gamma\delta} = 0, \quad 0 \leq \alpha,\ \beta,\ \gamma,\ \delta \leq 4.$$

Therefore N is flat along M. And we complete the proof of Theorem. □

6 $Spin^c$ Structures, Seiberg–Witten Equations

In the proof of Positive Mass Conjecture for 5-dimensional Lorentzian manifolds, we have to assume spacelike hypersurface is spin so as to ensure the global existence of hypersurface spinor S. For general spacelike hypersurface, S may not exist globally, but we shall show that $S \otimes L^{\frac{1}{2}}$ does exist golbally where L is some U_1 line bundle. This is $Spin^c$ structure.

Firstly, we investigate some basic facts about $Spin^c$ structure. Let M be an orientable 4-dimensional Riemannian manifold. The group $Spin(4)$ is the double covering of $SO(4)$ with covering map $\xi\ :Spin(4) \longrightarrow SO(4)$. The group $Spin^c(4)$ is defined as

$$Spin^c(4) = Spin(4) \times_{Z_2} U_1$$

which we identify $(a,\ -b) \sim (-a,\ b)$ in $Spin(4) \times U_1$. Note that $Spin^c(4)$ is a double covering of $SO(4) \times U_1$. And the sequence

$$\begin{array}{rcl} 0 \longrightarrow Z_2 \longrightarrow Spin^c(4) & \xrightarrow{\xi^c} & SO(4) \times U_1 \longrightarrow 0 \\ [(a,\ b)] & \longmapsto & (\xi(a),\ b^2) \end{array} \tag{6.1}$$

is exact. where $Z_2 \subset Spin^c(4)$ is generated by $[(-1,\ 1)] = [(1,\ -1)]$.

Let $(U, \{g_{\alpha\beta}\})$, $(U, \{r_{\alpha\beta}\})$ be cocycles represent $F(M)$ and P_{U_1}. The liftings of $g_{\alpha\beta}$ and $r_{\alpha\beta}$ in $Spin^c(4)$ are $\widetilde{g}_{\alpha\beta}$ and $r^{\frac{1}{2}}_{\alpha\beta}$ respectively, where $\widetilde{g}_{\alpha\beta}$ is the lifting of $g_{\alpha\beta}$ from $SO(4)$ to $Spin(4)$ by the covering map ξ. Define

$$\omega^c_{\alpha\beta\gamma} = \widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha} r^{\frac{1}{2}}_{\alpha\beta} r^{\frac{1}{2}}_{\beta\gamma} r^{\frac{1}{2}}_{\gamma\alpha}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. Since

$$\xi^c(\omega_{\alpha\beta\gamma}) = g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}r_{\alpha\beta}r_{\beta\gamma}r_{\gamma\alpha} = 1,$$

we see that $\{\omega^c_{\alpha\beta\gamma}\}$ represents a Z_2 cocycle.

The short exact sequence (6.2) yields a long exact sequence

$$H^1(M,\ Spin^c(4)) \xrightarrow{\xi^c_*} H^1(M,\ SO(4)) \oplus H^1(M,\ U_1) \xrightarrow{\rho} H^2(M,\ Z_2),$$

$$[\widetilde{g}_{\alpha\beta} r^{\frac{1}{2}}_{\alpha\beta}] \overset{\xi^c_*}{\longmapsto} [g_{\alpha\beta}] \oplus [r_{\alpha\beta}] \overset{\rho}{\longmapsto} [\omega^c_{\alpha\beta\gamma}] = \omega_2 + [r^{\frac{1}{2}}_{\alpha\beta\gamma}].$$

The exact sequence

$$0 \longrightarrow Z_2 \longrightarrow U_1 \xrightarrow{2} U_1 \longrightarrow 0$$

yields

$$H^1(M,\ U_1) \longrightarrow H^1(M,\ U_1) \longrightarrow H^2(M,\ Z_2),$$

$$[r^{\frac{1}{2}}_{\alpha\beta}] \longmapsto [r_{\alpha\beta}] \longmapsto [r^{\frac{1}{2}}_{\alpha\beta\gamma}].$$

Hence, $\rho([g_{\alpha\beta}],\ [r_{\alpha\beta}]) = \omega_2(M) + \widetilde{c}_1(L)$, where $\widetilde{c}_1(L)$ is the *mod* 2 reduction of $c_1(L) \in H^2(M,\ Z)$.

The $Spin^c(4)$ structure over TM or T^*M is that $\{w^c_{\alpha\beta\gamma} = 1\}$,i.e., $\{w^c_{\alpha\beta\gamma}\}$ represents a cocycle in Céch cohomology. This gives $\omega_2(M) + \widetilde{c}_1(L) = 0$ and $S \otimes L^{\frac{1}{2}}$ is globally defined, where $S = (U,\ \{\widetilde{g}_{\alpha\beta}\})$ is local spinor bundle and $L^{\frac{1}{2}} = (U,\ \{r^{\frac{1}{2}}_{\alpha\beta}\})$ is square root of some U_1 line bundle. The distinct $Spin^c$ structures are in $1-1$ correspondence with element $H^1(M,\ Z_2) \oplus 2H^2(M,\ Z)$ since $Spin^c$ structure is composed by the *Spin* structure and the square root of line bundle.

Any orientable 4-dimensional Riemannian manifold admits a $Spin^c$ structure [H-H]. Denote the associated bundle of principal $Spin^c(4)$ bundle as W with correspendent U_1 line bundle L, then

$$W = S \otimes L^{\frac{1}{2}}.$$

For any connection $d_A = d + A$ on L with iR-value connection 1-form A, the induced connection on $L^{\frac{1}{2}}$ is just $\widetilde{d}_A = d + \frac{1}{2}\ A$, its curvature $\widetilde{F}_A = \frac{1}{2}\ dA = \frac{1}{2}\ F_A$, where F_A is the curvature of d_A which is iR-value 2-form. The connection ∇_A on W is contributed by the connection ∇ on S given in §3 together with a connection $\widetilde{d}_A$ on $L^{\frac{1}{2}}$. Explicitly, write $\phi = \sigma \otimes \varepsilon \in S \otimes L^{\frac{1}{2}}$, then

$$\nabla_A \phi = \nabla \sigma \otimes \varepsilon + \sigma \otimes \widetilde{d}_A \varepsilon.$$

We can define the Dirac operator D_A related to A. In a local coframe $\{e^i\}$ of M, $D_A = e^i.\nabla_{Ai}$. Under the action of $\bigwedge^2$ on spinors defined by (3.10), we have the following Weitzenböck formula [L-M].

$$D_A^* D_A = D_A^2 = \nabla_A^* \nabla_A + \frac{R}{4} + \frac{1}{2}\ F_A., \tag{6.2}$$

where R is the scalar curvature of M, $F_A.$ denotes Clifford multiplication by the 2-form F_A.

If M is an orientable spacelike hypersurface (not spin) in N. The choice of timelike covector e^0 also yields a diagram

$$\begin{array}{ccc} Spin^c(4) & \xrightarrow{\hat{\alpha}} & HU(1,1) \times_{Z_2} U_1 \\ \downarrow & & \downarrow \\ SO(4) \times U_1 & \xrightarrow{\alpha} & SO(4,1) \times U_1 \end{array}$$

and allow us to define the hypersurface spinors – also denoted $W-$ for $HU(1,1)\times_{Z_2} U_1$ structure along M. Moreover, $W = W^+ \oplus W^-$, $W^\pm = S^\pm \otimes L^{\frac{1}{2}}$ and $S^\pm$ are locally half hypersurface spinor bundles and L is some U_1 line bundle defined on M. Extend L with its connection d_A onto the neighborhood of M in N parallelly. This new d_A together with $\widetilde{\nabla}$ defines a connection $\widetilde{\nabla}_A$ on W. For this hypersurface spinor^c bundle, there are all the correspondent facts shown in §§3 – 4. The hypersurface Dirac operator $\widetilde{D}_A$ related to A can also be defined by the second connection $\widetilde{\nabla}_A$ on W. The relation to the usual D_A is still

$$\widetilde{D}_A\phi = D_A\phi + \frac{H}{2}e^0.\phi \tag{6.3}$$

for $\phi \in \Gamma(W)$, where $H = \sum h_{ii}$ is the mean curvature. Also we have the Weitzenböck formulas

$$\begin{aligned}\widetilde{D}_A^*\widetilde{D}_A = \widetilde{D}_A^2 &= \nabla_A^*\nabla_A + \frac{1}{4}\,(R+H^2) - \frac{1}{2}\,\nabla_i He^0.e^i. + \frac{1}{2}\,F_A. &(6.4)\\ &= \widetilde{\nabla}_A^*\widetilde{\nabla}_A + \frac{1}{2}\,(T_{00} + T_{0i}e^0.e^i.) + \frac{1}{2}\,F_A. &(6.5)\end{aligned}$$

Now we derive the Unique Continuation Property for this $\widetilde{D}_A$ (In fact, we show it for operator $D_A + V$, where D_A is the Dirac operator for $Spin^c(4)$ structure of M, V is either bounded covector/vector, or bounded function on M.).

Theorem 6.1 *(Carleman's type inequality, [B-W], §8) Let M be an orientible 4-dimensional manifold. D_A is the Dirac operator acts on spinors for $Spin^c$ structure. Let $B_p(r)$ and $B_p(R)$ be two balls in M centered at $p \in M$ radii r and $R(> r)$ respectively. Denote $N_{r,R}$ $= B_p(R) - B_p(r)$, $\rho(x) = dist(x,p)$. There are sufficiently small R_0, sufficiently large T_0 such that for any $R < R_0$, and $t > T_0$, the following inequality holds for any spinor $\phi|_{\partial N_{r,R}} = 0$,*

$$t\int_{N_{r,R}} e^{t(R-\rho)^2}|\phi|^2 \le C\int_{N_{r,R}} e^{t(R-\rho)^2}|D_A\phi|^2. \tag{6.6}$$

Where C is constant depending only on R_0, T_0 and the geometries of $B_p(R_0)$.

Theorem 6.2 *(Unique Continuation Property) Let M be an orientible connected smooth 4- dimensional manifold. D_A is the Dirac operator acts on spinor space W for $Spin^c$ structure. Let operator $P = D_A + V$ where $m = \sup_M |V| < \infty$. For any $\phi \in \Gamma(W)$ such that $P(\phi) = 0$ and $\phi = 0$ on some open set $\Omega \subset M$, we have $\phi \equiv 0$ on M.*

Proof. We just show that for any $q \in \partial\Omega$, $\phi(x)$ can be zero-extented from q. Choose a positive real r sufficiently small and a point $p \in V$ at a distance r from q such that the ball

$B_p(r) \in V$. Choose $R > r$ satisfies the conditions of Theorem 6.1 and $B_p(R) \cap (M - \Omega) \neq 0$. Let

$$\chi = \begin{cases} 1 & \{x \in N_{r,R} : \rho(x) \leq (1-2\delta)R\}; \\ 0 & \{x \in N_{r,R} : \rho(x) > (1-\delta)R\}, \end{cases}$$

for some small $\delta > 0$. And let $u(x) = \chi(x)\phi(x)$. Since $P(\phi) = 0$, $\phi|_{\partial\Omega} = 0$,

$$D_A(u) = d\chi.\ \phi - \chi V \phi.$$

But $supp(d\chi) \subset N_{(1-2\delta)R,(1-\delta)R}$, $u|_{\partial N_{r,R}} = 0$, the Carleman inequality (6.6) gives

$$\begin{aligned} (t - cm^2) e^{\frac{t}{4}R^2} \int_{N_{r,\frac{R}{2}}} |\phi|^2 &\leq (t - cm^2) \int_{N_{r,R}} e^{t(R-\rho)^2} |u|^2 \\ &\leq \int_{N_{r,R}} e^{t(R-\rho)^2} |d\chi\phi|^2 \\ &\leq e^{4t\delta^2 R^2} \int_{N_{r,R}} |\phi|^2 \end{aligned}$$

for $t \gg m^2$. Take $\delta = \frac{1}{4\sqrt{2}}$, we obtain

$$\int_{N_{r,\frac{R}{2}}} |\phi|^2 \leq (t - cm^2)^{-1} e^{-\frac{R^2}{4}t} \int_{N_{r,R}} |\phi|^2.$$

Let $t \to \infty$ in the above inequality, we obtain

$$\phi(x)|_{N_{r,\frac{R}{2}}} = 0.$$

Thus we get the zero-extension of ϕ from $q \in \partial\Omega$. Hence $\phi(x) \equiv 0$ and we complete the proof. □

Corollary 6.1 *Unique Continuation Property holds for the hypersurface Dirac operator $\widetilde{D}_A$ with $\sup_M |H| < \infty$.*

Proposition 6.1 *If the spacelike hypersurface M is not maximal ('maximal' means $H \equiv 0$), then any 'half' spinor solution $\widetilde{D}_A \phi = 0$ is trivial.*

Proof. If there is such spinor $\phi^+ \in \Gamma(W^+)$. Then

$$0 = \widetilde{D}_A \phi^+ = D_A \phi^+ + \frac{H}{2} \phi^+ \ \in \Gamma(W^-) \oplus \Gamma(W^+).$$

Hence $D_A \phi^+ = 0$ and $H \cdot \phi^+ = 0$. By the assumption, there is an open set Ω such that $H \neq 0$ on Ω, therefore $\phi^+ = 0$ on Ω. Then the Unique Continuation property gives that $\phi^+ \equiv 0$. Similarly, there is no such nontrivial $\psi^- \in \Gamma(W^-)$. □

Now we introduce the Seiberg-Witten equations. For any $\phi,\ \psi \in \Gamma(W)$, denote

$$\sigma(\phi,\ \psi) = \frac{1}{4} \sum_{i,j} \langle e^i.e^j.\phi,\ \psi\rangle e^i \wedge e^j. \tag{6.7}$$

Since

$$\langle e^i.e^j.\phi,\ \psi\rangle = -\langle \phi,\ e^i.e^j.\psi\rangle = -\overline{\langle e^i.e^j.\psi,\ \phi\rangle},$$

we see that $\sigma(\phi,\ \psi)$ is an imaginary 2-form.

Denote

$$e^I = e^1 \wedge e^2 + e^3 \wedge e^4, \quad e^J = e^1 \wedge e^3 + e^4 \wedge e^2, \quad e^K = e^2 \wedge e^3 + e^4 \wedge e^1.$$

If $\phi^+ \in \Gamma(W^+)$, then (3.11) implies that

$$\sigma(\phi^+, \phi^+) = \frac{1}{2}\left(\langle i\phi^+, \phi^+\rangle e^I + \langle j\phi^+, \phi^+\rangle e^J + \langle k\phi^+, \phi^+\rangle e^K\right). \tag{6.8}$$

Direct computation shows that

$$\langle F^+.\phi^+,\ \phi^+\rangle = 2\langle F^+,\ \sigma(\phi^+, \phi^+)\rangle \tag{6.9}$$

for any self-dual 2-form F^+. And

$$\langle \sigma(\phi^+, \phi^+).\phi^+,\ \phi^+\rangle = |\phi^+|^4. \tag{6.10}$$

The Seiberg-Witten equations on 4-dimensional Riemannian manifolds are

$$\left\{ \begin{array}{lcl} D_A\phi^+ & = & 0 \\ F_A^+ & = & \sigma(\phi^+, \phi^+). \end{array} \right. \tag{6.11}$$

It was shown by Witten [W2] that when M is compact, without boundary,

$$\begin{aligned} & \int_M |D_A\phi^+|^2 + \frac{1}{2} |F_A^+ - \sigma(\phi^+, \phi^+)|^2 \\ = & \int_M |\nabla_A\phi^+|^2 + \frac{R}{4}\, |\phi^+|^2 + \frac{1}{4}\, |\phi^+|^4 + \frac{1}{2}\, |F_A^+|^2. \end{aligned} \tag{6.12}$$

Hence (6.11) is just the minimum of the right-side functional of (6.12).

We generalize them to the 4-dimensional spacelike hypersurface $M \subset N$ by

$$\left\{ \begin{array}{lcl} \widetilde{D}_A\phi & = & 0 \\ F_A^+ & = & \sigma(\phi^+, \phi^+), \end{array} \right. \tag{6.13}$$

where $\phi \in \Gamma(W)$ and ϕ^+ is the positive part of ϕ. We call them the hypersurface Seiberg-Witten equations. When M is compact and without boundary, we have

Theorem 6.3

$$\begin{aligned}
&\int_M |\widetilde{D}_A\phi|^2 + \frac{1}{2}\,|F_A^+ - \sigma(\phi^+,\phi^+)|^2 \\
= &\int_M |\nabla_A\phi^+|^2 + \frac{R+H^2}{4}\,|\phi^+|^2 + \frac{1}{4}\,|\phi^+|^4 + \frac{1}{2}\,|F_A^+|^2 + \\
&\int_M |D_A\psi^-|^2 - Re\langle dH.\psi^-,\ \phi^+\rangle + \frac{H^2}{4}\,|\psi^-|^2, \qquad (6.14)
\end{aligned}$$

where $\phi = \phi^+ + \psi^-$. Hence (6.13) is the minimum of the right-side functional of (6.14). Moreover, (6.13) and (6.14) are rescaled invariance of the Lorentzian metric $\widetilde{g}$ on N.

Proof. The first part is the direct consequence of (6.2), (6.3) and (6.12). In fact,

$$\begin{aligned}
\int_M |\widetilde{D}_A\phi|^2 &= \int_M |D_A\phi^+ - \frac{H}{2}\,\psi^-|^2 + \int_M |D_A\psi^- + \frac{H}{2}\,\phi^+|^2 \\
&= \int_M |D_A\phi^+|^2 + \frac{H^2}{4}\,|\psi^-|^2 + |D_A\psi^-|^2 + \frac{H^2}{4}\,|\phi^+|^2 + \\
&\quad \int_M HRe\langle D_A\psi^-,\ \phi^+\rangle - HRe\langle D_A\phi^+,\ \psi^-\rangle \\
&= \int_M |D_A\phi^+|^2 + \frac{H^2}{4}\,|\phi^+|^2 + \\
&\quad \int_M |D_A\psi^-|^2 - Re\langle dH.\psi^-,\ \phi^+\rangle + \frac{H^2}{4}\,|\psi^-|^2.
\end{aligned}$$

Hence it follows in terms of (6.12). For the second part, we just show that

$$E(A,\ \phi^+,\ \psi^-,\ \widetilde{g}) = E(A,\ \lambda^{-1}\phi^+,\ \lambda^{-1}\psi^-,\ \lambda^2\widetilde{g}),$$

where $E(A,\ \phi^+,\ \psi^-,\ \widetilde{g})$ denotes the right-side of (6.14). Let the rescaled metric $\widetilde{g}_1 = \lambda^2\widetilde{g}$ (λ is constant). With respect to the new metric, $\nabla_A^{\tilde{g}_1} = \nabla_A^{\tilde{g}}$, $(\nabla_A^{\tilde{g}_1})_{\tilde{e}_i} = \lambda^{-1}(\nabla_A^{\tilde{g}})_{e_i}$, $D_A^{\tilde{g}_1} = \lambda^{-1}D_A^{\tilde{g}}$, and the pointwise norm on a 1-form gets multiplied by λ^{-1} and that of a 2-form by λ^{-2}. Hence we have

$$|\nabla_A(\lambda^{-1}\phi)|^2_{\widetilde{g}_1} = \lambda^{-4}|\nabla_A\phi|^2_{\widetilde{g}}, \qquad |F_A|^2_{\widetilde{g}_1} = \lambda^{-4}|F_A|^2_{\widetilde{g}}$$

Recall the well-known facts [E] that the scalar curvature of the rescaled metric is given by $R_1 = \lambda^{-2}R$, and the mean curvature is given by $H_1 = \lambda^{-1}H$, thus, $d_1H_1.\psi^- = \lambda^{-2}dH.\psi^-$. This shows that the integrand of the functional (6.14) scales with the factor λ^{-4}. Since the volume form of the rescaled metric is given by $dvol_{\widetilde{g}_1} = \lambda^4 dvol_{\widetilde{g}}$, it follows that the whole integral remains unchanged. This proves the theorem. □

Remark 6.1 *When $H \equiv 0$, (6.14) reduces to (6.12).*

Finally, we derive a vanishing theorem which generalizes an observation by Witten that any L^2 (or finite energy) solution of (6.11) on R^4 has vanishing ϕ^+ [W2]. Our theorem corresponds to the infinite energy solution. Firstly, we recall a theorem of Cheng-Yau [C-Y1],

Theorem 6.4 *Suppose M is a complete Riemannian manifold with Ricci $\geq -K$ $(K \geq 0)$. Let u be a C^2-solution of $\triangle u \geq f(u)$ where f is a non-negative lower semi-continuous function and*

$$\lim_{t\to\infty} \inf \frac{f(t)}{g(t)} > 0$$

for some positive continous function g non-decreasing on some intergral $[a, \infty)$ with

$$\int_b^\infty (\int_a^t g(\tau)d\tau)^{-\frac{1}{2}}dt < \infty$$

for some b. Then $\sup u$ exists and is a zero of function f.

In terms of this theorem, we easily obtain

Theorem 6.5 *Suppose M is a complete 4-dimensional Riemannian manifold with Ricci $\geq -K$ $(K \geq 0)$. If the scalar curvature $R \geq 0$, then any C^2-solution of (6.11) on M has vanishing ϕ^+.*

Proof. Let ϕ^+ be a C^2-solution of (6.11), then the assumption on $R \geq 0$ and Weitzenböck formula (6.2) give that

$$\triangle|\phi^+|^2 \geq \frac{1}{2}\,|\phi^+|^4.$$

Let $u = |\phi^+|^2$, $f(t) = \frac{1}{2}\,t^2$ and $g(t) = 3t^2 + 1$, g is increasing and

$$\lim_{t\to\infty} \inf \frac{f(t)}{g(t)} = \frac{1}{6}.$$

Furthormore,

$$\int_0^\infty (\int_0^t g(\tau)d\tau)^{-\frac{1}{2}}dt = \int_0^\infty \frac{dt}{\sqrt{t^3+t}} < \infty.$$

Hence the above theorem gives that $f(\sup u) = 0$, which implies $\phi^+ \equiv 0$. □

7 The Mean Curvature of Spacelike Hypersurfaces

In this section, we always assume M is a spacelike hypersurface in $N^{4,1}$. We shall show that, in general, the mean curvature of M must vanish at some points when the nontrivial solutions of $\widetilde{D}\phi = 0$ exist. It is at least two aspects of importance: The nonexistence of constant mean curvature hypersurfaces which are much interesting in general relativity; On the other hand, the compactness of the moduli space of hypersurface Seiberg-Witten equations (6.13) fails since it depends on the nonzero lower bound of $|H|$, see [Z], such a bound does not exist as shown as follows.

Proposition 7.1 *Suppose $M \subset N$ is a compact spacelike hypersurface. If there exists a nontrivial solution ϕ for the hypersurface Dirac equation $\widetilde{D}_A\phi = 0$, moreover, if M has boundary, $\phi|_{\partial M} = 0$, then*

$$\int_M H|\phi|^2 = 0. \tag{7.1}$$

Therefore H must vanish at some points $p \in M$. Particularly, any such M with constant mean curvature is maximal.

Proof. Let $\phi = \phi^+ + \psi^-$ be the nontrivial solution of $\widetilde{D}_A\phi = 0$. Then

$$\begin{aligned} 0 &= \widetilde{D}_A\phi = D_A(\phi^+ + \psi^-) + \frac{H}{2}\, e^0.(\phi^+ + \psi^-) \\ &= (D_A\phi^+ - \frac{H}{2}\,\psi^-) + (D_A\psi^- + \frac{H}{2}\,\phi^+) \ \in \Gamma(W^-) \oplus \Gamma(W^+). \end{aligned}$$

Therefore

$$D_A\phi^+ - \frac{H}{2}\,\psi^- = 0, \qquad D_A\psi^- + \frac{H}{2}\,\phi^+ = 0.$$

Since D_A is self-adjoint under the metric $\langle\ ,\ \rangle$,

$$\int_M \langle D_A\phi^+,\ \psi^-\rangle = \int_M \langle \phi^+,\ D_A\psi^-\rangle.$$

We obtain

$$\int_M H(|\phi^+|^2 + |\psi^-|^2) = 0.$$

Hence it follows. □

Proposition 7.2 *Suppose $M \subset N$ is a complete, noncompact spacelike hypersurface without boundary. If there exists a nontrivial L^2 solution for the hypersurface Dirac operator $\widetilde{D}_A\phi = 0$, then H must vanish at some points $p \in M$. Particularly, any such M with constant mean curvature is maximal.*

Proof. Let $\phi = \phi^+ + \psi^-$ be the nontrivial L^2 solution of $\widetilde{D}_A\phi = 0$. Let cut-off function χ be 1 in the ball $B(R)$, zero outside the ball $B(2R)$ and $|d\chi| \leq \frac{C}{R}$ for some $C > 0$. Then Lemma 4.2 gives that

$$\begin{aligned} 0 &= \int_{B(2R)} (\langle D_A\phi^+,\ \chi\psi^-\rangle - \langle \phi^+,\ D_A(\chi\psi^-)\rangle). \\ &= \int_{B(2R)} (\chi(-H)(|\phi^+|^2 + |\psi^-|^2) - \langle \phi^+,\ d\chi.\psi^-\rangle). \end{aligned}$$

If H does not change sign, then

$$\left|\int_{B(R)} H|\phi|^2\right| \leq \left|\int_{B(2R)} \chi H|\phi|^2\right| \leq \int_{B(2R)} |d\chi||\phi^+||\psi^-| \leq \frac{C}{2R}\int_M |\phi|^2.$$

Hence the proposition follows in taking $R \to \infty$. □

Proposition 7.3 *Suppose $M \subset N$ is a complete, noncompact spacelike hypersurface without boundary. On the $L^2(S)$ spinor space, we have $Ker(\widetilde{D}) = Ker(\widetilde{D}^2)$.*

Proof. We just need to show that for any $\phi \in L^2(S)$ such that $\widetilde{D}^2\phi = 0$, we must have $\widetilde{D}\phi = 0$. Let cut-off function χ be 1 in the ball $B(R)$, zero outside the ball $B(2R)$ and $|d\chi| \leq \frac{C}{R}$ for some $C > 0$. Since

$$\widetilde{D}(\chi^2\widetilde{D}\phi) = 2\chi d\chi.\widetilde{D}\phi + \chi^2 D\widetilde{D}\phi + \frac{H}{2}e^0.\chi^2\widetilde{D}\phi = 2\chi d\chi.\widetilde{D}\phi,$$

then Lemma 4.2 gives that

$$\begin{aligned} 0 &= \int_{B(2R)} \chi^2|\widetilde{D}_A\phi|^2 - \langle\phi,\ \widetilde{D}(\chi^2\widetilde{D}\phi)\rangle \\ &= \int_{B(2R)} \chi^2|\widetilde{D}_A\phi|^2 - 2\langle\phi,\ \chi d\chi.\widetilde{D}\phi\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{B(2R)} \chi^2|\widetilde{D}_A\phi|^2 &\leq \frac{2C}{R}\int_{B(2R)} \chi|\phi||\widetilde{D}\phi| \\ &\leq \frac{2C^2}{R^2}\int_{B(2R)} |\phi|^2 + \frac{1}{2}\int_{B(2R)} \chi^2|\widetilde{D}_A\phi|^2. \end{aligned}$$

Taking $R \to \infty$ gives $\widetilde{D}\phi = 0$. □

Theorem 7.1 *Suppose $M \subset N$ is a compact spin spacelike hypersurface without boundary and satisfies*

$$R + H^2 \geq 2|\nabla H|, \tag{7.2}$$

where R, H are the scalar and mean curvatures of M respectively. If the strict inequality in (7.2) holds at some points in M, then the hypersurface Dirac equation

$$\widetilde{D}\phi = 0 \tag{7.3}$$

has only trivial solution; Otherwise (7.3) has nontrivial solutions which are covariant constants, hence $H = 0$ and $R = 0$, i.e., M must be maximal with zero scalar curvature.

Proof. Obviously, Lemma 4.2 implies $Ker(\widetilde{D}) = Ker(\widetilde{D}^2)$ when M has no boundary. Consider the equation

$$\widetilde{D}^2\phi = \nabla^*\nabla\phi + K.\phi = 0, \tag{7.4}$$

under the boundary condition

$$\int_M \phi = \eta, \tag{7.5}$$

where $K = \frac{1}{4}(R + H^2) - \frac{1}{2}\nabla_i H e^0.e^i.$, $\phi = (\phi_1, \phi_2) \in S$. Under the assumption (7.2), any solution ϕ of (7.4) has

$$0 = \int_M |\nabla\phi|^2 + \langle \phi, K.\phi \rangle \geq \int_M |\nabla\phi|^2 + \frac{1}{4}(R + H^2 - 2|\nabla H|)|\phi|^2, \tag{7.6}$$

and the strict inequality holds if $R + H^2 > 2|\nabla H|$ at some points and ϕ is not zero identically. Hence the first part of the theorem follows via the unique continuation property of $\widetilde{D}$. For the second part, (7.6) implies $\widetilde{D}^2$ has trivial kernel when $\eta = 0$. Hence there exists a unique smooth ϕ such that $\widetilde{D}^2\phi = 0$ and $\int_M \phi = 1$. This ϕ is the nontrivial solution of the equation $\widetilde{D}\phi = 0$ and $\nabla\phi = 0$. In terms of this solution, (4.1) gives $H \equiv 0$, thus $R = 0$ follows from the assumption. □

Remark 7.1 *If M is an asymptotically flat spacelike hypersurface and (7.2) holds on M, then for any L^2 solution of $\widetilde{D}\phi = 0$, Weitzenböck formula gives $\Delta|\phi|^2 \geq 0$. Since $|\phi|^2 \in L^1$, a theorem of Li ([L], Theorem 1) implies $\phi \equiv 0$.*

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